Test 3 Numerical Mathematics 2
January 26, 2023

Duration: 90 minutes.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark of this test. Use of a calculator is allowed.

1. Consider the inner product $(f,g) = \int_{-1}^{1} |x| f(x) g(x) dx$.
   (a) [2.5] Derive the degree 0, 1 and 2 orthogonal polynomials associated with this inner product. *Hint:* Occurring integrals can be computed more easily when exploiting symmetry.
   (b) [1.5] Which integral, related to the above inner product, can be approximated by a Gauss rule based on the orthogonal polynomials of part a? Use the quadratic orthogonal polynomial of the previous part to determine the associated Gauss rule.
   (c) [0.5] According to the theory, how does the 'degree of exactness' of a Gauss rule relate to the degree of the orthogonal polynomial used to define the rule. Check explicitly whether the rule derived in b is in accordance with the theory.

2. Consider the function $f(x)$ on [0,1] given by

$$f(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}], \\ 1 & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

   The aim of the exercise is to make a Chebyshev expansion of this function.
   (a) [1] Which property holds in general for the error of the polynomial of best approximation to a general continuous function $f(x)$? Make a sketch of the given function $f(x)$ and prove that the best linear approximation on [0,1] is given by $x + 1/4$. Make also a sketch of the error.
   (b) [1] Show that the Chebyshev expansion of $f(x)$ can be written as $C_n(x) = \sum_{k=0}^{n} a_k T_k(-1+2x)$ where for general f and g the following inner product holds $(f,g) = \int_0^1 1/\sqrt{1-(-1+2x)^2} f(x) g(x) dx$. Also give the expression for a_k.
   (c) [0.8] Using b it can be shown that (you do not have to do the computation) $a_0 = 1 - \frac{1}{\pi}$ and $a_1 = \frac{1}{2}$. Add a sketch of $C_1(x)$ to the figures of part a. State the theorem of de la Vallée-Poussin and apply it to derive an as sharp as possible lower bound for the the minimax error? Also give an upper bound based on the sketched error.
   (d) [0.5] In this case, will $C_n(x)$ converge faster than any positive power of $1/n$? Explain your answer.
   (e) i. [0.5] Show that $(f, T_k(-1+2x)) = \frac{1}{2} \int_0^{\pi} \cos(k\theta) f(\frac{1+\cos(\theta)}{2}) d\theta$.
   ii. [0.2] What will be the result of the previous part if also $f(x) = T_k(-1+2x)$?
   iii. [0.5] Show that $a_0 = 1 - \frac{1}{\pi}$.